Inequality 2

15 October 2023 11:35

HomeWork :- $(a, b, x \in \mathbb{R})$

1> $|ab| = |a||b|$

2> $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$ with $b \neq 0$

3> $|x| \leq b \iff -b \leq x \leq b$

4> $\big||a| - |b|\big| \leq |a - b|$

$$\left|\frac{a}{b}\right| = \begin{cases} \frac{a}{b} & \text{if } \frac{a}{b} \geq 0 \\ -\frac{a}{b} & \text{if } \frac{a}{b} < 0 \end{cases}$$

$\frac{a}{b} \geq 0 \rightarrow a \geq 0$ and $b > 0$; $\rightarrow a < 0$ and $b < 0$

$\frac{a}{b} < 0 \rightarrow a \geq 0$ and $b < 0$; $\rightarrow a < 0$ and $b > 0$

$$\frac{|a|}{|b|} = \begin{cases} \frac{a}{b} & \text{if } a \geq 0 \text{ and } b > 0 \\ -\frac{a}{b} & \text{if } a \geq 0 \text{ and } b < 0 \\ -\frac{a}{b} & \text{if } a < 0 \text{ and } b > 0 \\ \frac{a}{b} & \text{if } a < 0 \text{ and } b < 0 \end{cases}$$

$$\Rightarrow \left|\frac{a}{b}\right| = \frac{|a|}{|b|}$$

General Form of Triangle Inequality :-

For real numbers $x_1, x_2, \ldots, x_n$ we get,

$$|x_1 + x_2 + \cdots + x_n| \leq |x_1| + |x_2| + \cdots + |x_n|$$

Proof:- $y = x_2 + \cdots + x_n$

Proof:- $y_2 = x_2 + \cdots + x_n$

$|x_1 + y_2| \le |x_1| + |y_2| \le |x_1| + |x_2| + |y_3|$

$y_3 = x_3 + \cdots + x_n$

$|x_2 + y_3| \le |x_2| + |y_3|$

and this goes on ..

$\Rightarrow |x_1 + x_2 + \cdots + x_n| \le |x_1| + |x_2| + \cdots + |x_n|$

Q) $x^2 + xy + y^2 \geq 0 \quad \forall\, x, y \in \mathbb{R}$. Prove it

Ans:- WLOG, $x > y$,

Case 1 :- $x \geq 0, y > 0 \Rightarrow LHS \geq 0$

Case 2 :- $x < 0, y < 0 \Rightarrow LHS \geq 0$

Case 3 :- $x > 0, y < 0 \Rightarrow LHS = \quad xy \in \mathbb{R}^-,\ x^2, y^2 \in \mathbb{R}^+$

$\Rightarrow |x^2| > |xy| \Rightarrow x^2 + xy > 0 \Rightarrow LHS \geq 0$

Case 4 :- Similarly.

Q) For real numbers a, b, c prove that,

$$|a| + |b| + |c| - |a+b| - |b+c| - |c+a| + |a+b+c| \geq 0$$

Ans:- WLOG, $|a| \geq |b| \geq |c| > 0$

$\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$

$\left|\frac{c}{a}\right| \le 1, \quad \left|\frac{b}{a}\right| \le 1$

$$\left|\frac{a}{a}\right| + \left|\frac{b}{a}\right| + \left|\frac{c}{a}\right| - \left|\frac{a+b}{a}\right| - \left|\frac{b+c}{a}\right| - \left|\frac{c+a}{a}\right| + \left|\frac{a+b+c}{a}\right|$$

$$= 1 + \left|\frac{b}{a}\right| + \left|\frac{c}{a}\right| - \left|1 + \frac{b}{a}\right| - \left|\frac{b}{a} + \frac{c}{a}\right| - \left|1 + \frac{c}{a}\right| + \left|1 + \frac{b}{a} + \frac{c}{a}\right|$$

$\left|\frac{c}{a}\right| \le 1 \Rightarrow 1 + \frac{c}{a} \geq 0 \Rightarrow \left|1 + \frac{c}{a}\right| = 1 + \frac{c}{a}$

$\left|\frac{b}{a}\right| \leq 1 \Rightarrow 1+\frac{b}{a} \geqslant 0 \Rightarrow \left|1+\frac{b}{a}\right| = 1+\frac{b}{a}$

$$= \left|\frac{b}{a}\right| + \left|\frac{c}{a}\right| - \left|\frac{b}{a}+\frac{c}{a}\right| + \left(-1-\frac{b}{a}-1-\frac{c}{a}+1\right) + \left|1+\frac{b}{a}+\frac{c}{a}\right|$$

$$= \underbrace{\left|\frac{b}{a}\right| + \left|\frac{c}{a}\right| - \left|\frac{b}{a}+\frac{c}{a}\right|}_{\geqslant 0} \underbrace{- \left(1+\frac{b}{a}+\frac{c}{a}\right) + \left|1+\frac{b}{a}+\frac{c}{a}\right|}_{\geqslant 0}$$

triangle inequality $\qquad$ $|x|-x$ form

So LHS $\geqslant 0$

HomeWork

Q> $a, b \in \mathbb{R}$ and $0 < a \leq b \leq 1$. Prove that $0 \leq \frac{b-a}{1-ab} \leq 1$

Q> $a, b \in \mathbb{R}$ and $0 < a \leq b \leq 1$. Prove that

$$0 \leq \left(\frac{a}{1+b} + \frac{b}{1+a}\right) \leq 1$$

Q> $a, b \in \mathbb{R}$ and $0 < a \leq b \leq 1$. Prove that,

$$0 \leq ab^2 - ba^2 \leq \frac{1}{4}$$

Q> Prove that $\frac{m}{n} < \sqrt{2}$ iff $\sqrt{2} < \frac{m+2n}{m+n}$. m, n are positive integers

Ans:- $x < \sqrt{2}$, $x > 0$

$1+\frac{1}{1+x}$ is a decreasing function when $x > 0$

As $x < \sqrt{2}$ so lower bound value is at $x = \sqrt{2}$.

$$\Leftrightarrow 1+\frac{1}{1+x} > 1+\frac{1}{1+\sqrt{2}} = \frac{\sqrt{2}+1+1}{1+\sqrt{2}} = \frac{\sqrt{2}(1+\sqrt{2})}{1+\sqrt{2}} = \sqrt{2}$$

so $1+\frac{1}{1+x} > \sqrt{2}$

If $x = \frac{m}{n}$, $1 + \frac{1}{1+\frac{m}{n}} > \sqrt{2}$

$$\Leftrightarrow 1 + \frac{n}{m+n} > \sqrt{2} \Leftrightarrow \frac{m+2n}{m+n} > \sqrt{2}$$